Graphs with (Edge) Disjoint Links in Every Spatial Embedding

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Abstract: We describe some characteristics of graphs with edge and vertex disjoint pairs of links in every spatial embedding. In particular, we show that the smallest complete graph that contains pairs of non-splittable links in every spatial embedding is K_7. Further we find a large set of minimal graphs that have pairs of non-splittable links sharing no vertices in every spatial embedding. We show that triangle-Y exchanges and vertex splittings are disjoint link preserving operations. We find regular graphs of arbitrarily high genus that have a linkless embedding. We show, by counter-example, that it is not true that a three-component non-splittable link exists in every spatial embedding of $K_{3,3,3}$.

Background: *Definitions:* Throughout this paper we will take an *embedded graph* to mean a graph embedded in three-space, where all of our embeddings are tame. A *link* is a finite set of disjoint cycles embedded in three-space. A link is said to be *non-splittable* if there exists no sphere in three-space disjoint from the link that separates components of the link. We must extend the definition of a link in order to consider cycles within a graph. We say an embedded graph is *linked* provided it contains a set of disjoint cycles that form a non-splittable link. A graph is said to be *intrinsically linked* if, in every spatial embedding, the graph is linked.

A *vertex splitting* of a vertex v in a graph is achieved by replacing v with two vertices v' and v", adding the edge v'v" and connecting a subset of the edges which were initially incident to v to v' and the rest of the edges which were initially incident to v to v". A graph G is considered to be an *expansion* of a graph H if G can be obtained from H by vertex splittings. Let Q be a graph and let s be a vertex of Q with only three incident edges. Let P be obtained from Q by deleting s and adding an edge between each pair of vertices connected to the edges initially

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incident to s. We say that P is obtained from Q by a *Y-triangle* exchange and that Q is obtained from P by a *triangle-Y* exchange.

Topics: It has previously been shown by Conway and Gordon [4] and Sachs [9] that every spatial embedding of the complete graph K_6 contains at least one pair of disjoint cycles that form a non-splittable link. We say that K_6 is intrinsically linked. In 1998, Chris Campbell [3] proved that in every spatial embedding of K_{10} there exists a pair of non-splittable links that share no edges, but may share vertices. We say that K_{10} has the edge disjoint linking property. A graph is said to have the *edge disjoint linking property* if, in every spatial embedding, there exists a pair of non-splittable links that share no edges. We have looked further into this question and have found that the smaller complete graph K_9, K_8, and K_7 all have the edge disjoint linking property.

In 1993, Robertson, Seymour and Thomas [8] proved that a graph is intrinsically linked if and only if it contains one of the seven Petersen graphs, which can be obtained from K_6 through triangle-Y exchanges, Y-triangle exchanges, or an expansion of the graph. In an attempt to find an analog of this theorem for the vertex disjoint linking property, we found different types of graphs that contain non-splittable links that share no vertices and no edges in every spatial embedding. A graph is said to have the *vertex disjoint linking property* if, in every spatial embedding, there exists a pair of non-splittable links that share no vertices. Some examples of graphs that have this property are disjoint copies of Petersen graphs, expansions of disjoint copies of Petersen graphs, and a finite set of graphs that we have found with this property that do not contain at least two disjoint copies of Petersen graphs. Using graph expansion methods such as triangle-Y exchanges and Y-triangle exchanges, we have found this set to be much larger than the set of Petersen graphs.

The effects that certain graph operations have on the edge disjoint linking property and the vertex disjoint linking property were also studied. It was shown by Sachs [9] that vertex splittings and triangle-Y exchanges preserve intrinsic linking in a graph. We have found similar results for the vertex and edge disjoint linking properties. In particular, both vertex splittings and triangle-Y exchanges preserve the vertex disjoint linking property, but vertex splitting does not preserve the edge disjoint linking property. We conjecture that triangle-Y exchanges do not preserve the edge disjoint linking property.